INTERNATIONAL JOURNAL OF MULTIDISCIPLINARY FUTURISTIC DEVELOPMENT

Application of Graph Theory to Digital Topology and Image Analysis

Dr. Sujata Tiwari
Department of Mathematics, P.K. University, Shivpuri, Madhya Pradesh, India

* Corresponding Author: **Dr. Sujata Tiwari**

Article Info

P-ISSN: 3051-3618
E-ISSN: 3051-3626
Impact Factor (RSIF): 8.31
Volume: 07
Issue: 02
Received: 26-04-2026
Accepted: 28-05-2026
Published: 30-06-2026
Page No: 45-50

Abstract
Graph theory provides a powerful mathematical framework for representing and analyzing the structural and topological properties of digital images. A digital image can naturally be modeled as a graph in which pixels or voxels constitute vertices and adjacency relationships constitute edges. Such representations allow fundamental topological properties, including connectivity, connected components, paths, boundaries, cycles, holes, and structural continuity, to be studied using discrete mathematical techniques. The present paper examines the application of graph theory to digital topology and image analysis, with particular emphasis on pixel and voxel adjacency, connected-component analysis, image segmentation, boundary detection, skeletonization, region adjacency graphs, medical image analysis, and topological feature extraction. The paper adopts a theoretical and analytical research methodology and integrates classical graph-based digital topology with contemporary developments in computational topology and topological data analysis. Particular attention is given to 4- and 8-adjacency in two-dimensional images and 6-, 18-, and 26-adjacency in three-dimensional images. The study also examines the integration of graph structures with persistent homology and machine-learning techniques. It is observed that graph-based representations transform complex spatial relationships into computationally manageable discrete structures while preserving selected connectivity information. Recent advances further demonstrate that topological features can complement conventional pixel-intensity-based and deep-learning approaches in image classification and segmentation. The paper concludes that the integration of graph theory, digital topology, persistent homology, and artificial intelligence provides a promising mathematical framework for structurally informed image analysis.

DOI: https://doi.org/10.54660/IJMFD.2026.7.2.45-50

Keywords: Graph Theory, Digital Topology, Image Analysis, Digital Images, Connectivity, Image Segmentation, Adjacency Graph, Persistent Homology, Medical Imaging

1. Introduction

The rapid development of digital imaging technologies has created an increasing need for mathematical methods capable of analyzing the structural characteristics of images. Digital images are widely used in medicine, remote sensing, computer vision, industrial inspection, autonomous systems, biological research, security, and scientific visualization. Although an image is commonly represented as an array of numerical intensity or color values, many image-analysis problems depend not merely on individual pixel values but on relationships among neighboring pixels and regions.

Graph theory provides an appropriate framework for representing these relationships. A graph is mathematically expressed as

$$G = (V, E),$$

where Vrepresents a set of vertices and Erepresents edges connecting selected pairs of vertices. In digital image analysis, pixels or voxels can be represented by vertices, while spatial or functional relationships among them can be represented by edges.
Digital topology extends topological concepts such as connectedness, boundaries, holes, surfaces, and continuity to discrete digital spaces. Classical digital-topology research established theoretical foundations for image-processing operations including connected-component labeling, border following, contour filling, and thinning. The graph-based approach is particularly important because adjacency relations can be used to determine whether digital points belong to the same connected object.

The relationship can be summarized as

Digital Image → Pixels/Voxels → Vertices
→ Adjacency Edges → Graph
→ Topological Analysis.

Modern image-analysis methods have expanded beyond elementary graph connectivity. Graph cuts, region adjacency graphs, skeleton graphs, simplicial complexes, persistent homology, graph neural networks, and topology-aware deep-learning methods provide increasingly sophisticated tools for preserving and extracting structural information.
This paper examines the theoretical relationship between graph theory, digital topology, and image analysis and discusses both traditional and contemporary applications.

2. Objectives of the Study
The major objectives of the study are:
1. To examine the relationship between graph theory and digital topology.
2. To explain the graph representation of two-dimensional and three-dimensional digital images.
3. To analyze the role of graph connectivity and adjacency in digital image analysis.
4. To examine applications of graph theory in connected-component analysis, segmentation, boundary detection, and skeletonization.
5. To investigate graph-theoretic applications in medical and scientific image analysis.
6. To examine the integration of digital topology with persistent homology and machine learning.
7. To identify the major advantages, limitations, and future research directions of graph-based digital image analysis.

3. Research Methodology
The present study adopts a theoretical, descriptive, and analytical research methodology. It synthesizes established mathematical concepts from graph theory, digital topology, computational topology, and image analysis.
The conceptual framework begins with graph representations of digital images and then analyzes connectivity, adjacency, components, paths, boundaries, and cycles. Classical concepts are subsequently connected with modern approaches involving region adjacency graphs, persistent homology, and topology-aware machine learning.
The study is conceptual rather than experimental. Consequently, no artificial accuracy values, image datasets, or statistical results are reported. Instead, the analysis focuses on mathematical relationships and established applications documented in the literature up to 2025.

4. Graph-Theoretic Representation of Digital Images
A digital image may be represented as a graph

$$G = (V, E).$$

Each pixel $p_i$can correspond to a vertex

$$v_i \in V.$$

An edge

$$e_{ij} = (v_i, v_j)$$

is introduced when pixels $p_i$and $p_j$satisfy a specified adjacency relationship.
For a weighted image graph, a weight may also be assigned:

$$w_{ij} = f(p_i, p_j),$$

where $f$measures similarity, dissimilarity, intensity difference, spatial distance, color difference, or another relationship.
For example, a simple intensity-based weight may depend on

$$| I(p_i) - I(p_j) |,$$

where $I(p)$denotes pixel intensity.
Thus, a digital image is transformed from a rectangular numerical array into a mathematical network that explicitly represents relationships among image elements.

5. Digital Topology and Adjacency
Adjacency is fundamental to digital topology because a digital space is discrete. In ordinary Euclidean topology, points may be connected through continuous paths. In a digital image, connectivity must be defined according to neighborhood relationships.

5.1 Four-Adjacency
For a pixel

$$p = (x, y),$$

its four-neighborhood is

$$N_4(p) = \{(x, +1y), (x, -1y), (x, y + 1), (x, y - 1)\}.$$

Two pixels are 4-adjacent when one lies immediately above, below, to the left, or to the right of the other.

5.2 Eight-Adjacency
The eight-neighborhood additionally includes diagonal positions:

$$N_8(p) = N_4(p) \cup \{(x, +1y + 1), (x, +1y - 1), (x, -1y + 1), (x, -1y - 1)\}.$$

Thus,

$$N_4(p) \subseteq N_8(p).$$

The selection between 4- and 8-adjacency affects the connectedness of digital objects.
Consider the binary arrangement

$$\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$$

The two foreground pixels are disconnected under 4-adjacency but connected under 8-adjacency.
This demonstrates that digital connectivity is dependent on the selected adjacency convention.

6. Three-Dimensional Digital Topology

Three-dimensional medical and scientific images consist of **voxels** rather than pixels. Common neighborhood structures include 6-, 18-, and 26-adjacency.
Under **6-adjacency**, a voxel is connected to neighbors sharing a face.
Under **18-adjacency**, face-sharing and edge-sharing neighbors are included.
Under **26-adjacency**, face-, edge-, and corner-sharing neighbors are considered.
Hence,

$$N_6(p) \subseteq N_{18}(p) \subseteq N_{26}(p).$$

These adjacency structures are particularly important in CT, MRI, microscopy, geological imaging, and three-dimensional object reconstruction.

7. Graph Connectivity and Connected Components

Let

$$G = (V, E)$$

represent a binary digital object.
A path between vertices v_0 and v_n is a sequence

$$v_0, v_1, \dots, v_n$$

such that

$$(v_i, v_{i+1}) \in E$$

for every consecutive pair.
The graph is connected when every pair of vertices is joined by a path.
A **connected component** is a maximal connected subgraph.
If an image contains several isolated objects, the corresponding graph contains several connected components:

$$G = C_1 \cup C_2 \cup \cdots \cup C_k.$$

Connected-component analysis is therefore one of the clearest applications of graph theory to image processing.
It can be used for identifying separate cells, characters, objects, lesions, particles, buildings, roads, or other image regions.

8. Connected-Component Labeling

Connected-component labeling assigns a distinct label to each connected foreground region of a binary image.
The graph-theoretic procedure can be expressed as:

$$\text{Image} \to \text{Adjacency Graph} \to \text{Graph Traversal} \to \text{Connected Components} \to \text{Object Labels}.$$

Algorithms such as breadth-first search and depth-first search can traverse the image graph and determine which vertices belong to each component.

If

$$c(G)$$

denotes the number of connected components, then $c(G)$ can correspond to the number of distinct objects under the chosen adjacency rule.
Applications include cell counting, document analysis, defect detection, satellite-image interpretation, and object recognition.

9. Graph Theory in Image Segmentation

Image segmentation divides an image into meaningful regions. Graph theory provides a natural representation of this task.

Consider

$$G = (V, E, W),$$

where W represents edge weights.
Vertices may represent individual pixels, superpixels, or regions, while edge weights represent similarity.
If two neighboring pixels have similar intensity or texture, their connecting edge may indicate strong similarity. Large differences may indicate probable boundaries.
The segmentation problem can then be interpreted as dividing the graph into subsets

$$V_1, V_2, \dots, V_k$$

such that vertices within a subset are strongly related and vertices belonging to different subsets are comparatively weakly related.
Graph-cut and related partitioning techniques formalize this general idea.

10. Region Adjacency Graphs

Instead of representing every pixel individually, an image can first be divided into regions. Each region becomes a vertex.
A **Region Adjacency Graph (RAG)** may be written as

$$G_R = (V_R, E_R),$$

where each $v_i \in V_R$ represents an image region and an edge exists when two regions share a boundary.
This significantly reduces graph complexity.
Attributes associated with vertices may include:

- average intensity,
- color,

- texture,
- area,
- shape, and
- spatial position.

Edges may encode boundary strength or region similarity.
Region adjacency graphs are useful for hierarchical segmentation, region merging, scene interpretation, and object extraction.

11. Boundary Detection
Boundaries separate image objects from their surroundings. Graph theory provides a structural approach to detecting such boundaries.
If vertices represent pixels and edges represent neighborhood relations, significant changes in edge weights can indicate transitions between image regions.
A boundary may therefore be associated with a collection of graph edges separating two vertex subsets.
The general representation is

$$\text{Region A} \mid \text{Boundary} \mid \text{Region B}.$$

Digital topology is important because boundary extraction should preserve structural consistency. An incorrectly removed or added pixel may create or destroy a connection or hole.
Graph-based boundary analysis is particularly useful when topology must be preserved during image transformation.

12. Skeletonization and Thinning
Skeletonization reduces a digital object to a simplified representation while attempting to preserve its essential topology.
For example, a thick branching object can be transformed into a thin network representing its structural centerlines.
Graphically,

$$\text{Original Object} \rightarrow \text{Thinning} \rightarrow \text{Skeleton} \rightarrow \text{Graph}.$$

Vertices of the resulting skeleton may represent endpoints, junctions, or important structural points, while edges represent branches.
A desirable thinning process should preserve important properties such as:

$$\text{Number of Components},$$
$$\text{Connectivity},$$

and, where relevant,

$$\text{Number of Holes}.$$

Skeletonization is useful in handwriting recognition, fingerprint analysis, vascular imaging, road extraction, plant-root analysis, and biological morphology.

13. Graph Cycles and Holes
Graph theory also provides information related to cyclic structures.
For a finite connected graph,

$$\beta_1 = |E| - |V| + 1$$

gives the dimension of its cycle space.
More generally, if the graph has $c$connected components,

$$\beta_1 = |E| - |V| + c.$$

This quantity describes independent graph cycles.
However, graph cycles and holes in a two-dimensional image should not automatically be regarded as identical. A graph representing pixel adjacency may contain cycles that do not correspond one-to-one with geometric holes.
For richer topological analysis, cubical or simplicial complexes and homology provide a more rigorous framework.

14. From Graph Theory to Topological Data Analysis
Modern image analysis increasingly extends graph representations using **Topological Data Analysis (TDA)**.
Persistent homology examines how topological features appear and disappear as a scale or threshold parameter changes.

For an image filtration

$$K_0 \subseteq K_1 \subseteq \cdots \subseteq K_n,$$

topological features can be tracked across successive levels.
Important quantities include:

$$\beta_0 = \text{number of connected components},$$
$$\beta_1 = \text{number of one-dimensional holes},$$

and, in suitable three-dimensional settings,

$$\beta_2 = \text{number of enclosed voids}.$$

Persistent homology distinguishes short-lived features, which may represent noise, from structures persisting over a broader range of thresholds.
The resulting information can be represented using persistence diagrams or barcodes.

15. Application to Medical Image Analysis
Medical imaging is one of the most important application areas for graph-theoretic and topological image analysis.
CT, MRI, ultrasound, microscopy, histopathology, and retinal images contain anatomical structures whose connectivity and shape can carry diagnostically important information.
Graph-based representations are useful for analyzing:

- blood vessels,
- neuronal structures,
- airways,
- cell networks,
- tumors,
- retinal vascular systems,
- cardiac structures, and
- tissue organization.

For example, a vascular image can be converted into a graph in which branching points become vertices and vessel segments become edges. Graph properties can subsequently quantify branching structure and connectivity.
Topological information can complement conventional intensity and texture information because two objects with similar pixel distributions may have substantially different structural organizations.

16. Persistent Homology in Contemporary Medical Imaging

Persistent homology has increasingly been incorporated into medical image-analysis pipelines.

Instead of relying only on local image intensity, persistent homology extracts multi-scale structural characteristics. These features can subsequently be integrated with conventional machine-learning or deep-learning models.

A typical pipeline can be represented as

$$\text{Medical Image} \rightarrow \text{Filtration} \rightarrow \text{Persistent Homology} \rightarrow \text{Topological Features} \rightarrow \text{Classifier/Segmenter}.$$

Recent research has demonstrated the integration of persistent-homology information with convolutional neural networks and Transformer architectures for medical image classification. Such methods seek to capture connected components and loops that conventional feature learning may not explicitly preserve.

Persistent homology has also been investigated for mitochondrial morphology and other biological-image structures, illustrating its usefulness for quantitatively describing complex spatial organization.

17. Topology-Aware Deep Learning

Deep-learning segmentation models may achieve strong pixel-level performance while still producing structurally incorrect predictions.

For example, a predicted blood vessel may contain an artificial break:

$$\text{True vessel: } A - - - - - - - - B$$

while a segmentation model may produce

$$A - - - - \quad - - - - B.$$

Only a small number of pixels may be incorrect, but the topology has changed substantially because one connected structure has become two.

Topology-aware learning attempts to penalize these structural errors.

Contemporary methods therefore incorporate topological loss functions or persistent-homology-based constraints into neural-network training.

This leads to the general framework:

$$\text{Pixel Accuracy} + \text{Topological Accuracy} \rightarrow \text{Structurally Faithful Segmentation}.$$

Such an approach is especially valuable for vessels, neurons, roads, airways, and other network-like structures.

18. Graph Theory and Machine Learning

Graph theory is also increasingly integrated with machine learning through Graph Neural Networks (GNNs).

If an image-derived structure is represented as

$$G = (V, E, X),$$

where Xcontains vertex features, a GNN can learn representations by exchanging information between neighboring vertices.

Image graphs can therefore combine:

$$\text{Spatial Features} + \text{Graph Connectivity} + \text{Topological Features}.$$

This approach is particularly promising for biological and medical structures in which relational information is important.

Brain-connectivity research, for example, has combined graph convolutional networks with persistent-homology information to analyze disease-related alterations in connectivity.

19. Limitations and Challenges

Despite its advantages, graph-based digital topology has several limitations.

The first is **graph size**. A high-resolution image may contain millions of pixels, resulting in a graph with millions of vertices and many more adjacency relationships.

Second, the choice of adjacency strongly influences the results. Four- and eight-adjacency may produce different connectivity interpretations in the same two-dimensional image.

Third, noise can create false vertices, edges, components, or holes.

Fourth, ordinary graphs mainly capture pairwise relationships and may not fully represent higher-dimensional topology.

Fifth, persistent-homology calculations and topology-aware learning can increase computational complexity.

Finally, combining topological information with deep-learning architectures requires careful balancing because improved pixel-level accuracy does not automatically imply improved topological correctness.

20. Major Findings

The theoretical analysis leads to several major findings.

Graph theory provides a natural discrete framework for digital topology because pixels and voxels can be represented as vertices and adjacency relations as edges.

Connectivity is one of the strongest links between graph theory and digital images. Connected-component analysis allows separate image objects to be identified computationally.

Graph representations are highly useful in segmentation, boundary detection, region analysis, and skeletonization.

The choice of adjacency is mathematically significant and must be explicitly specified.

Ordinary graph cycles should not automatically be interpreted as image holes; algebraic-topological structures provide more rigorous tools for higher-dimensional features.

Persistent homology extends conventional graph connectivity by analyzing components, holes, and voids across multiple thresholds.

Recent developments indicate increasing integration between topology and deep learning, particularly where structurally faithful segmentation is required.

Graph theory therefore functions both as an independent image-analysis framework and as a foundation for more advanced topological and machine-learning approaches.

21. Future Scope

The future of graph-theoretic digital topology is likely to involve deeper integration with artificial intelligence.

One important direction is the development of topology-aware neural networks that simultaneously optimize conventional prediction accuracy and structural correctness.

Another direction involves combining persistent homology with graph neural networks for medical and biological image classification.

Efficient topological algorithms for very large three-dimensional images will also be important.

Further research can investigate dynamic or temporal image graphs, explainable topology-aware AI, multi-parameter persistent homology, uncertainty-aware topology, and hybrid graph-transformer architectures.

Applications are particularly promising in vascular imaging, neuroscience, histopathology, autonomous navigation, remote sensing, material science, and cellular imaging.

22. Conclusion

Graph theory provides a powerful mathematical framework for digital topology and image analysis. By representing pixels, voxels, image regions, or structural landmarks as vertices and their relationships as edges, complex images can be transformed into discrete mathematical structures suitable for rigorous analysis.

The study demonstrates that graph connectivity forms the foundation of connected-component labeling and plays an important role in segmentation, boundary analysis, skeletonization, and object recognition. Four- and eight-adjacency in two dimensions and 6-, 18-, and 26-adjacency in three dimensions provide alternative neighborhood structures whose selection directly influences digital connectivity.

The integration of graph theory with computational topology considerably expands these capabilities. Persistent homology makes it possible to study connected components, holes, and voids across multiple scales, while topology-aware machine learning seeks to prevent structurally incorrect predictions that may not be adequately penalized by conventional pixel-level measures.

Recent research through 2025 indicates a movement toward hybrid approaches in which graph structures, topological descriptors, persistent homology, and artificial intelligence are jointly employed. Such methods are particularly valuable for images containing complex structural networks such as blood vessels, neurons, mitochondrial systems, roads, and anatomical regions.

It can therefore be concluded that graph theory serves as an essential bridge between discrete image representation and topological reasoning. Its combination with digital topology and modern artificial intelligence provides substantial scope for developing more structurally accurate, mathematically interpretable, and computationally effective image-analysis systems.

References

1. Barmak JA. Algebraic topology of finite topological spaces and applications. Springer; 2011.
2. Bian C, Xia N, Xie A, Cong S, Dong Q. Adversarially trained persistent homology based graph convolutional network for disease identification using brain connectivity. IEEE Trans Med Imaging. 2024;43(1):503-516.
3. Chung YM, Hu CS, Sun E, Tseng HC. Morphological multiparameter filtration and persistent homology in mitochondrial image analysis. PLoS One. 2024;19(9):e0310157.
4. Diestel R. Graph theory. 5th ed. Springer; 2017.
5. Edelsbrunner H, Harer J. Computational topology: an introduction. American Mathematical Society; 2010.
6. Ghrist R. Elementary applied topology. Createspace Independent Publishing Platform; 2014.
7. Gonzalez RC, Woods RE. Digital image processing. 4th ed. Pearson; 2018.
8. Hatcher A. Algebraic topology. Cambridge University Press; 2002.
9. Kaczynski T, Mischaikow K, Mrozek M. Computational homology. Springer; 2004.
10. Khalimsky E, Kopperman R, Meyer PR. Computer graphics and connected topologies on finite ordered sets. Topol Appl. 1990;36(1):1-17.
11. Kong TY, Rosenfeld A. Digital topology: introduction and survey. Comput Vis Graph Image Process. 1989;48(3):357-393.
12. Kong TY, Rosenfeld A, editors. Topological algorithms for digital image processing. Elsevier; 1996.
13. Kong TY, Kopperman R, Meyer PR. A topological approach to digital topology. Am Math Mon. 1991;98(10):901-917.
14. Abiad A, Arenas A, Backhausz Á, Balogh J, Banerji CRS, Barbarossa S, et al. Hypergraphs and simplicial complexes in focus: a roadmap for future research in higher-order interactions. J Phys Complex. 2026;7(2):022501.
15. Kozlov D. Combinatorial algebraic topology. Springer; 2008.
16. Peng Y, Wang H, Sonka M, Chen DZ. PHG-Net: persistent homology guided medical image classification. In: Proceedings of the IEEE/CVF Winter Conference on Applications of Computer Vision; 2024. p. 7583-7592.
17. Rosenfeld A. Digital topology. Am Math Mon. 1979;86(8):621-630.
18. West DB. Introduction to graph theory. 2nd ed. Prentice Hall; 2001.
19. Zomorodian A. Topology for computing. Cambridge University Press; 2005.
20. Brito-Pacheco D, Giannopoulos P, Reyes-Aldasoro CC. Persistent homology in medical image processing: a literature review. medRxiv. 2025. Advance online publication.

How to Cite This Article

Tiwari S. Application of graph theory to digital topology and image analysis. Int J Math Found Dev. 2026;7(2):45-50. doi:10.54660/IJMFD.2026.7.2.45-50.